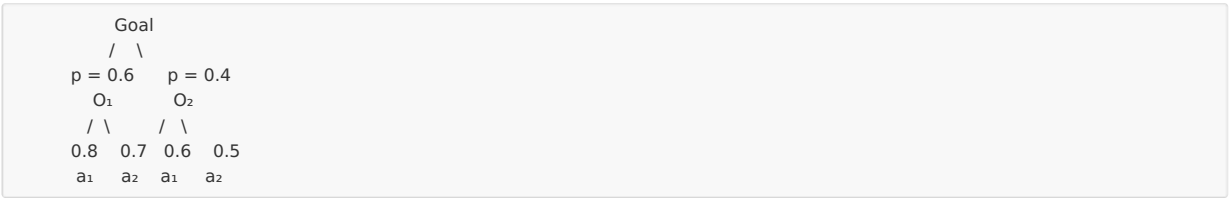
Unnormalized vs Normalized

Consider a simple model with **two objectives** (O_1 and O_2) and **two alternatives** (a_1 and a_2).
Let's assume that we provided judgments for a_1 and a_2 wrt O_1 and O_2, and those judgments are in unnormalized mode (for instance, we used ratings), so that priorities of a_1 and a_2 do not add up to 1 wrt O_1 and O_2.



Here:

- For O_1: $p(a_1) + p(a_2) = 0.8 + 0.7 \neq 1$
- For O_2: $p(a_1) + p(a_2) = 0.6 + 0.5 \neq 1$

Now, let's see how we calculate the resulting (global) priorities of a_1 and a_2 in normalized and unnormalized modes.

1. Normalized mode:

First, we normalize priorities of a_1 and a_2 wrt each covering objective:

$$p(a1 \text{ wrt } O1)=0.8(0.8+0.7)=0.53$$
$$p(a2 \text{ wrt } O1)=0.7(0.8+0.7)=0.47$$

$$\sum=1\sum=1$$

$$p(a1 \text{ wrt } O2)=0.6(0.6+0.5)=0.55$$
$$p(a2 \text{ wrt } O2)=0.5(0.6+0.5)=0.45$$

$$\sum=1\sum=1$$

After that, we perform regular synthesis:

$$p(a1)=0.53\cdot0.6+0.55\cdot0.4=0.538$$
$$p(a_1) = 0.53 \cdot 0.6 + 0.55 \cdot 0.4 = 0.538 p(a1) = 0.47\cdot0.6+0.45\cdot0.4=0.462$$

Since all clusters were normalized, global priorities are also normalized: **0.538 + 0.462 = 1**

2. Unnormalized mode:

In this case, we skip the normalization step and go straight to synthesis:

$$p(a1)=0.8\cdot0.6+0.6\cdot0.55=0.72 p(a_1) = 0.8 \cdot 0.6 + 0.6 \cdot 0.4 = 0.72$$
$$p(a2)= 0.7\cdot0.6+0.5\cdot0.4=0.62$$

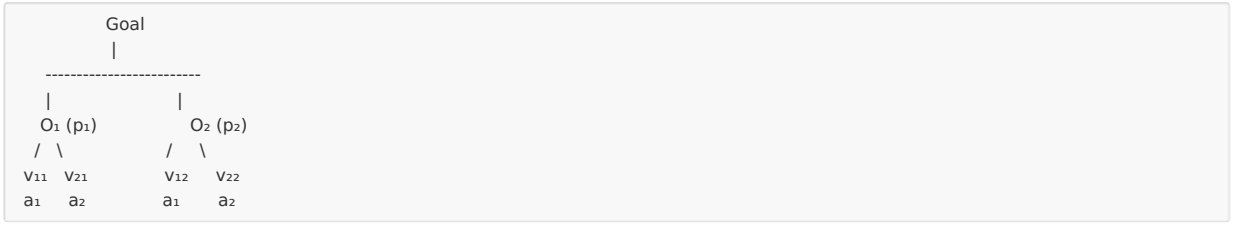
As we can see, the sum of global priorities does not add up to 1: $0.72+0.62=1.34$

If we normalize global unnormalized priorities, we will get the following values:

$$p(a1)=0.537 p(a_1) = 0.537$$
$$p(a2)= 0.463$$

Results are close, but not quite the same.

Let's check the general case and see if the results should or should not match up.



In normalized mode:

$$pN(a1)=p1 \cdot \frac{v_{11}}{v_{11} + v_{21}} + p2 \cdot \frac{v_{12}}{v_{12} + v_{22}}$$

(normalized priority wrt O_1 and wrt O_2)

In unnormalized mode after normalization:

$$pUN(a1)=\frac{p_1 v_{11} + p_2 v_{12}}{p_1 v_{11} + p_2 v_{12} + p_1 v_{21} + p_2 v_{22}}$$

(denominator is normalization)

If results are the same in normalized mode and unnormalized after normalization, then:

$$pN(a1)=pUN(a1)$$

That is:

$$p1 \cdot \frac{v_{11}}{v_{11} + v_{21}} + p2 \cdot \frac{v_{12}}{v_{12} + v_{22}} = \frac{p1 v_{11} + p2 v_{12}}{p1 v_{11} + p2 v_{12} + p1 v_{21} + p2 v_{22}}$$

Let's assume $p1=p2=0.5$ to simplify calculations:

$$\frac{v_{11}}{v_{11} + v_{21}} + \frac{v_{12}}{v_{12} + v_{22}} = \frac{v_{11} + v_{12}}{v_{11} + v_{12} + v_{21} + v_{22}}$$

$$\frac{v_{11}(v_{12} + v_{22}) + v_{12}(v_{11} + v_{21})}{(v_{11} + v_{21})(v_{12} + v_{22})} = \frac{v_{11} + v_{12}}{v_{11} + v_{12} + v_{21} + v_{22}}$$

$$(v_{11}v_{12} + v_{11}v_{22} + v_{12}v_{11} + v_{12}v_{21})(v_{11} + v_{12} + v_{21} + v_{22}) = (v_{11} + v_{12})(v_{11}v_{22} + v_{12}v_{21} + v_{21}v_{22}) = (v_{11} + v_{12})(v_{11}v_{22} + v_{12}v_{21} + v_{21}v_{22})$$

$$v_{12}v_{21}^2 + v_{11}v_{22}^2 + 2v_{11}v_{12} + 2v_{11}v_{12}v_{22} + 2v_{11}v_{12}v_{21} + 2v_{11}v_{21}v_{22} + 2v_{12}v_{21}v_{22} - v_{11}^2 v_{12} - v_{12}^2 v_{11} - v_{12}v_{21}^2 - v_{11}^2 v_{22} - v_{11}v_{22}^2 - v_{12}v_{21}^2 - v_{11}v_{22}^2 - v_{11}^2 v_{12} - v_{12}^2 v_{11} - 2v_{11}v_{12}v_{21} - 2v_{11}v_{12}v_{22} = 0$$

$$v_{12}v_{21}^2 + v_{11}v_{22}^2 + v_{11}v_{12} + v_{12}v_{11} + 2v_{11}v_{12}v_{21} + 2v_{11}v_{12}v_{22} = 0$$

$$v_{12}v_{22}^2 + v_{11}^2 v_{12} + v_{12}^2 v_{11} + 2v_{11}v_{12}v_{21} + 2v_{11}v_{12}v_{22} = 0$$

This is true only if:

$$v_{11} = v_{12} = v_{21} = v_{22} = 0$$

Conclusion: We should **NOT** expect results in normalized mode to match results in unnormalized mode after normalization.